ISI(A)

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Content of the Syllabus

Unit-I

Calculus: Partial Differentiation, Euler's Theorem, Composite functions, Jacobian. Taylors and Maclaurine Series for one and two variables. Multiple Integrals, Change of order and Change of Variable. Area and volume using double and triple integrals. [15H]

Unit-II

Calculus: Indeterminate forms and L'Hospital's rule

Algebraic structures: Definition, elementary properties of algebraic structures, semigroup monoid, group, homomorphism, isomorphism and automorphism, congruence relations, subgroups, normal subgroups, cosets, Lagrange's theorem, cyclic groups. [15H]

Unit-III

Vector spaces : Vector Space, linear dependence of vectors, Basis, dimension; Linear transformations (maps), range and kernel of a linear map, rank and nullity, Inverse of a linear transformation, rank- nullity theorem(Without Proof) , composition of linear maps, Matrix associated with a linear map. Inner product spaces, Gram-Schmidt orthogonalization.

Differentiation and Integration Formulas

The **Differential Calculus** splits up an area into small parts to calculate the rate of change. The **Integral calculus** joins small parts to calculates the area or volume and in short, is the method of reasoning or calculation.

Differential Calculus Formulas:

Differentiation is a process of finding the derivative of a function. The derivative of a function is defined as y = f(x) of a variable x, which is the measure of the rate of change of a variable y changes with respect to the change of variable x.

These rules make the differentiation process easier for different functions such as trigonometric functions, logarithmic functions, etc. Here, a list of differential calculus formulas is given below:

- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}(fg) = fg' + gf'$
- $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$
- $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$
- $\frac{d}{dx}(\sin\ x) = \cos\ x$
- $\frac{d}{dx}(\cos\ x) = -\sin\ x$
- $\frac{d}{dx}(\tan\ x) = \sec^2\ x$
- $\frac{d}{dx}(\cot\ x) = -\csc^2\ x$
- $\frac{d}{dx}(\sec\ x) = \sec\ x \tan\ x$
- $\frac{d}{dx}(\csc\ x) = -\csc\ x \cot\ x$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \ln\ a$
- $\frac{d}{dx}\ln\ x = \frac{1}{x}$
- $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

Integral Calculus Formulas

Integration is the process of continuous addition and the variable "C" represents the constant of integration. Also, it helps to find the area under the curve of a function. There are certain important integral calculus formulas helps to get the solutions. The list of integral calculus formulas is given below:

- $\int a\,dx = ax + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$
- $\int a^x\,dx = \frac{a^x}{\ln a} + C\,;\, a > 0,\, a \neq 1$
- $\int \ln x\,dx = x\ln x - x + C$
- $\int \sin\ x\,dx = -\cos\ x + C$
- $\int \cos\ x\,dx = \sin\ x + C$
- $\int \tan\ dx = \ln|\sec\ x| + C\ or\ -\ln|\cos\ x| + C$
- $\int \cot\ x\,dx = \ln|\sin\ x| + C$
- $\int \sec\ x\,dx = \ln|\sec\ x + \tan\ x| + C$
- $\int \csc\ x\,dx = \ln|\csc\ x - \cot\ x| + C$
- $\int \sec^2\ x\,dx = \tan\ x + C$
- $\int \sec\ x\ \tan\ x\,dx = \sec\ x + C$
- $\int \csc^2\ x\,dx = -\cot\ x + C$
- $\int \tan^2\ x\,dx = \tan\ x - x + C$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
- $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left|\frac{x + \sqrt{a^2 + x^2}}{a}\right| + C$

Trigonometry Formulas

All the formulas of trigonometry chapter are provided here for students to help them solve problems quickly.

sin(−θ) = −sin θ
cos(−θ) = cos θ
tan(−θ) = −tan θ
cosec(−θ) = −cosecθ
sec(−θ) = sec θ
cot(−θ) = −cot θ

Product to Sum Formulas

sin x sin y = 1/2 [cos(x–y) − cos(x+y)]
cos x cos y = 1/2[cos(x–y) + cos(x+y)]
sin x cos y = 1/2[sin(x+y) + sin(x−y)]
cos x sin y = 1/2[sin(x+y) – sin(x−y)]

Sum to Product Formulas

sin x + sin y = 2 sin [(x+y)/2] cos [(x-y)/2]
sin x – sin y = 2 cos [(x+y)/2] sin [(x-y)/2]
cos x + cos y = 2 cos [(x+y)/2] cos [(x-y)/2]
cos x – cos y = -2 sin [(x+y)/2] sin [(x-y)/2]

Identities

sin2 A + cos2 A = 1
1+tan2 A = sec2 A
1+cot2 A = cosec2 A

Basic Trigonometric Formulas

cos (A + B) = cos A cos B – sin A sin B
cos (A – B) = cos A cos B + sin A sin B
sin (A+B) = sin A cos B + cos A sin B
sin (A -B) = sin A cos B – cos A sin B
tan(A+B) = [(tan A + tan B)/(1 – tan A tan B)]

tan(A-B) = [(tan A – tan B)/(1 + tan A tan B)]
cot(A+B) = [(cot A cot B – 1)/(cot B + cot A)]
cot(A-B) = [(cot A cot B + 1)/(cot B – cot A)]
sin2A = 2sinA cosA = [2tan A /(1+tan2A)]
cos2A = cos2A–sin2A = 1–2sin2A = 2cos2A–1= [(1-tan2A)/(1+tan2A)]
tan 2A = (2 tan A)/(1-tan2A)
sin3A = 3sinA – 4sin3A
cos3A = 4cos3A – 3cosA
tan3A = [3tanA–tan3A]/[1–3tan2A]

Q:- $u = lx + my$ and $v = mx - ly$

Show that

(i) $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}$

(ii) $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$

$\left(\frac{\partial u}{\partial x}\right)_y \rightarrow$

$u \rightarrow (x, y)$

$\downarrow$

$\frac{\partial u}{\partial x} \rightarrow y$ is const

$\left(\frac{\partial x}{\partial u}\right)_v$

$x \rightarrow u, v$

Sol:- $u \rightarrow (x, y)$ $\quad u = lx + my$ —— (1)

$v \rightarrow (x, y)$ $\quad v = mx - ly$ —— (2)

from (1) and (2)

$lx + \cancel{my} = u$ —— (1) $\times l$

$mx - \cancel{ly} = v$ —— (2) $\times m$

$(l^2+m^2)x = lu + mv \Rightarrow x = \frac{lu+mv}{l^2+m^2}$ —— (3)

$\cancel{lx} + my = u \quad \times m$

$\cancel{mx} \underset{+}{-} ly = v \quad \times l$

$y = \frac{mu - lv}{}$ —— (4)

$$(m^2+l^2)y = mu - lv \Rightarrow y = \frac{mu-lv}{l^2+m^2} \quad (4)$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial}{\partial x}(lx+my) = l \checkmark$$

$[u = lx+my$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{\partial}{\partial u}\left(\frac{lu+mv}{l^2+m^2}\right) = \frac{l}{l^2+m^2}$$

[using (3)

$$\therefore \quad (1) \quad \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = (l)\left(\frac{l}{l^2+m^2}\right) = \frac{l^2}{l^2+m^2} \quad \text{Ans}$$

Q:- $u = \operatorname{Cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{\frac{1}{2}}$ Then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u).$$

Sol:- Given $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{\frac{1}{2}}$

$$\operatorname{Cosec} u = \left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{\frac{1}{2}} = z \quad (\text{say})$$

i.e $$z = \left(\frac{x^{\frac{1}{2}}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{\frac{1}{2}} = \left[\frac{x^{1/2}\left(1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right)}{x^{1/3}\left(1+\left(\frac{y}{x}\right)^{\frac{1}{3}}\right)}\right]^{\frac{1}{2}} = \left[x^{\frac{1}{6}}\frac{\left(1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right)}{\left(1+\left(\frac{y}{x}\right)^{\frac{1}{3}}\right)}\right]^{\frac{1}{2}}$$

$$= x^{1/2}\left(\frac{1+(y/x)^{\frac{1}{2}}}{\ldots}\right)^{\frac{1}{2}}$$

$$= x^{1/2}\left(\frac{1+(y/x)^{2}}{1+\left(\frac{y}{x}\right)^{1/3}}\right)^{\frac{1}{2}}$$

$$= x^{\frac{1}{12}} \phi\left(\frac{y}{x}\right)$$

where $\phi\left(\frac{y}{x}\right) = \left[\frac{1+(y/x)^{2}}{1+\left(\frac{y}{x}\right)^{\frac{1}{3}}}\right]^{\frac{1}{2}}$

$\therefore$ Z is a homogenous function in x and y with degree $\frac{1}{12}$.

Then By Euler's Theorem,

$$x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = \frac{1}{12} Z$$

$$x\frac{\partial}{\partial x}(\operatorname{cosec} u) + y\frac{\partial}{\partial y}(\operatorname{cosec} u) = \frac{1}{12}\operatorname{cosec} u$$

$$x(-\operatorname{cosec} u \cot u)\frac{\partial u}{\partial x} + y(-\operatorname{cosec} u \cot u)\frac{\partial u}{\partial y} = \frac{1}{12}\operatorname{cosec} u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(\frac{1}{12}\cancel{\operatorname{cosec} u}\right)\left(\frac{-1}{\cancel{\operatorname{cosec} u}\cot u}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u \quad \text{——— (1)}$$

partially diff w.r.t x on both sides,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u \frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$= \left(\frac{-1}{12}\sec^2 u - 1\right)\frac{\partial u}{\partial x}$$

multiplying both sides by x

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{-1}{12}\sec^2 u - 1\right)\left(x\frac{\partial u}{\partial x}\right) \quad \text{——— } (2)$$

partially diff w.r.t. y of equation (1).

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{-1}{12}\sec^2 u\,\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = \frac{-1}{12}\sec^2 u\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = \left(\frac{-1}{12}\sec^2 u - 1\right)\frac{\partial u}{\partial y}$$

Multiplying both sides by y,

$$xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \left(\frac{-1}{12}\sec^2 u - 1\right)\left(y\frac{\partial u}{\partial y}\right) \quad \text{——— } (3)$$

Adding equation (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left(\frac{-1}{12}\sec^2 u - 1\right)\left[\left(x\frac{\partial u}{\partial x}\right) + \left(y\frac{\partial u}{\partial y}\right)\right]$$

$$= \left(\frac{-1}{12}\sec^2 u - 1\right)\left(\frac{-1}{12}\tan u\right)$$

$$= \frac{1}{12}\tan u\left(\frac{\sec^2 u}{12} + 1\right)$$

$$= \frac{1}{12}\tan u\left(\frac{\sec^2 u + 12}{12}\right)$$

$$= \frac{1}{12} \tan u \left(\frac{1+\tan^2 u + 12}{12} \right)$$

$$= \frac{1}{144} \tan u \left(13 + \tan^2 u\right) \quad = R.H.S$$

Proved

Q:-(1) If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 24$$

Q:-2. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Then prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{-\sin u \cos 2u}{4\cos^3 u}.$$

Q:-3. If $u = \tan^{-1}(y^2/x)$, Show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin 2u \sin^2 u.$$

Q:-4. Given $Z = x^n f_1\left(\frac{y}{x}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$.

Then prove that $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z.$

Q:-5. If $u = \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$

Then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(1) $z = x^y$ find out the 1st order partial derivative.

Sol: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\quad) = 2x$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\quad) = 0 + 2y = 2y$

find 2nd order partial derivative.

$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial y^2}$ ✓

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 0$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(2x) = 0$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(2y) = 2$